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On Electromagnetic Duality in Locally Supersymmetric $N=2$ Yang–Mills Theory [★]

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Abstract

We consider duality transformations in $N = 2$ Yang–Mills theory coupled to $N = 2$ supergravity, in a manifestly symplectic and coordinate covariant setting. We give the essential of the geometrical framework which allows one to discuss stringy classical and quantum monodromies, the form of the spectrum of BPS saturated states and the Picard–Fuchs identities encoded in the special geometry of $N = 2$ supergravity theories.

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1 Introduction

Recently, proposals for the quantum moduli space of $N = 2$ rigid Yang–Mills theories [1] have been given in terms of particular classes of genus r Riemann surfaces parametrized by r complex moduli[2], r being the rank for the gauge group G broken to $U(1)^r$ for generic values of the moduli. The effective action for such theories, with terms up to two derivatives, is described by $N = 2$ supersymmetric lagrangians of r abelian massless vector multiplets[3], whose dynamics is encoded in a holomorphic prepotential $F(X^A)$, function of the moduli coordinates X^A ($A = 1, \dots, r$). According to Seiberg and Witten [1] this effective theory has classical, perturbative and non perturbative duality symmetries which reflect on monodromy properties of certain holomorphic symplectic vectors $(X^A, F_A(X))$, eventually related to periods of holomorphic one-forms[1]

$$\omega = X^A \alpha_A + F_A \beta^A, \quad (1.1)$$

where α_A, β^A is a basis for the $2r$ homology cycles of a genus r Riemann surface. The Picard–Fuchs equations, satisfied by the holomorphic vector one-form $U_i = (\partial_i X^A, \partial_i F_A)$ ($i = 1, \dots, r$) can be regarded as differential identities for “rigid special geometry”[4]. To attach a particular algebraic curve to “rigid special geometry” is therefore equivalent to exactly compute the holomorphic data U_i and therefore to exactly reconstruct the effective action for the self interaction of the r massless gauge multiplets once the massive states, both perturbative and non perturbative, have been integrated out. Indeed it is a virtue of $N = 2$ supersymmetry that all the couplings in the effective Lagrangian, including 4-fermion terms, can be computed purely in terms of the holomorphic data. Quite remarkably the quantum monodromies dictate the monopole and dyon spectrum

of the effective theory [1,2] which turns out to be “dual” to non-perturbative instanton effects [5] in the original G -invariant microscopic theory[6]. In this paper we consider several issues in order to extend the approach pursued in the rigid case to the more challenging case of coupling a $N = 2$ Yang–Mills theory to gravity. In particular we shall include in the $N = 2$ supergravity theory a dilaton–axion vector multiplet which is an essential ingredient to describe effective $N = 2$ theories which come from the low energy limit of $N = 2$ heterotic string theories in four dimensions[7]. Another ingredient is the extension of the “classical monodromies” to $N = 2$ local supersymmetry. For rigid theories the classical metric is essentially the Cartan matrix of the group G and the classical monodromies are related to the Weyl group of the Cartan subalgebra of G [2]. For $N = 2$ supergravity theories coming from $N = 2$ heterotic

strings, the classical metric of the moduli space of the pure gauge sector is based on the homogeneous space $O(2, r)/O(2) \times O(r)$ [3][7][8][9] and the classical monodromies are related to the T -duality group $O(2, r; \mathbb{Z})$ which in particular is an invariance of the massive charged states[10]. This state of affair is quite analogous to the analysis performed by Sen and Schwarz[11] for the $N = 4$ heterotic string compactifications, in which case an exact quantum duality symmetry $SL(2, \mathbb{Z}) \times O(6, r; \mathbb{Z})$ was conjectured [11–15] and a resulting spectrum for BPS states with both electric and magnetic states was proposed. In the $N = 4$ theory the $SL(2, \mathbb{Z}) \times O(6, r; \mathbb{Z})$ symmetry, using general arguments [16,17], has a natural embedding in $Sp(2(6 + r); \mathbb{Z})$, acting on the $6 + r$ vector self-dual field strengths $\mathcal{F}_{\mu\nu}^{+A}$ and their “dual” defined through $G_{+A}^{\mu\nu} \equiv -i \frac{\delta \mathcal{L}}{\delta \mathcal{F}_{\mu\nu}^{+A}}$. In generic $N = 2$ theories, because of quantum corrections [18,19], we do not expect such factorized $S - T$ duality to occur anymore[4]. Indeed this can be argued with a pure supersymmetry argument, related to the fact that once the classical moduli space $O(2, r)/O(2) \times O(r)$ is deformed by quantum corrections, then the factorized structure with the dilaton degrees of freedom is lost and a non trivial moduli space, mixing the S and T degrees of freedom should emerge. This result is in fact a consequence of a theorem on “special geometry” [20,21] which asserts that the only factorized special manifolds are the $\frac{SU(1,1)}{U(1)} \times \frac{O(2,r)}{O(2) \times O(r)}$ series, which precisely describe the “classical moduli space” of $S - T$ moduli. Because of the coupling to gravity, the symplectic structure and identification of periods, coming from special geometry, is also remarkably different from rigid special geometry. Indeed the interpretation of (X^Λ, F_Λ) , $\Lambda = 0, 1, \dots, r + 1$ as periods of algebraic curves is no longer appropriate to genus r Riemann surfaces, as it can be seen from the Picard–Fuchs [22,23] equations and from the form of the metric $g_{i\bar{j}} = -\partial_i \partial_{\bar{j}} \log i(\bar{F}_A X^A - \bar{X}^A F_A)$ of the moduli space [24–28,23,29]. In fact special geometry is known to be appropriate to a particular class of complex manifolds (Calabi–Yau manifolds or their mirrors) and to describe the deformations of the complex structure [26][17]. It is therefore tempting to argue that the quantum moduli space including $S - T$ duality and its monodromies is related to 3-manifolds (or their mirrors) with $h_{(2,1)} = r + 1$.

The paper is organized as follows: In chapter 2 we give a resumé of rigid theories, also discussing duality for the fermionic sector and the physical significance of monodromies and geometrical data, such as the holomorphic tensor C_{ijk} , related to the gaugino anomalous magnetic moment. In chapter 3 we describe in detail the coupling to gravity, the extension of duality to the fermionic sector and the existence of symplectic bases which do not admit a prepotential function F , as it occurs in certain formulations of $N = 2$ supergravities coming from $N = 2$ heterotic strings.

In chapter 4 we discuss classical and quantum duality symmetries and give generic formulae for the spectrum of the BPS states and the “semiclassical formulae” when the non perturbative spectrum is computed in terms of the “classical periods”. The explicit expression for the $r = 1, 2$ cases are given as examples. The paper ends with some concluding remarks.

An expanded version of these results is contained in [30].

2 Résumé of rigid special geometry

2.1 Basics

$N = 2$ supersymmetric gauge theory on a group G broken to $U(1)^r$, with $r = \text{rank } G$, corresponds to a particular case of the most general $N = 1$ coupling of r chiral multiplets (X^A, χ^A) to r $N = 1$ abelian vector multiplets $(\mathcal{A}_\mu^A, \lambda^A)$ in which the Kähler potential K and the holomorphic kinetic term function $f_{AB}(X^A)$ are given by

$$\begin{aligned} K &= i(\overline{F}_A X^A - F_A \overline{X}^A) \quad , \quad (F_A = \partial_A F) \\ f_{AB} &= \partial_A \partial_B F \equiv F_{AB} \end{aligned} \tag{2.1}$$

in terms of the single prepotential $F(X)$ [3]. One can show that the Kähler geometry is constrained because the Riemann tensor satisfies the identity [24,4]

$$R_{A\overline{B}C\overline{D}} = -\partial_A \partial_C \partial_P F \frac{\partial}{\partial \overline{B}} \frac{\partial}{\partial \overline{D}} \frac{\partial}{\partial \overline{Q}} \overline{F} g^{P\overline{Q}} \quad , \tag{2.2}$$

with

$$g_{P\overline{Q}} = \partial_P \partial_{\overline{Q}} K = 2 \text{Im } \partial_P \partial_{\overline{Q}} F \quad . \tag{2.3}$$

The bosonic lagrangian has the form

$$\begin{aligned} \mathcal{L} &= g_{A\overline{B}} \partial_\mu X^A \partial_\mu \overline{X}^B + (g_{A\overline{B}} \lambda^{IA} \sigma^\mu \mathcal{D}_\mu \overline{\lambda}^{\overline{B}}_I + \text{h.c.}) \\ &+ \text{Im } (F_{AB} \mathcal{F}_{\mu\nu}^{-A} \mathcal{F}_{\mu\nu}^{-B}) + \mathcal{L}_{\text{Pauli}} + \mathcal{L}_{4\text{-fermi}} \quad , \end{aligned} \tag{2.4}$$

where A, B, \dots run on the adjoint representation of the gauge group G , $I = 1, 2$ and $\mathcal{F}_{\mu\nu}^{+A} = \mathcal{F}_{\mu\nu}^A - \frac{i}{2} \epsilon_{\mu\nu\rho\sigma} \mathcal{F}^{\rho\sigma}$ ($\mathcal{F}_{\mu\nu}^{-A} = \overline{\mathcal{F}}_{\mu\nu}^{+A}$). As we shall see, also $\mathcal{L}_{\text{Pauli}}$ and $\mathcal{L}_{4\text{-Fermi}}$ contain the function F and its derivatives up to the fourth.

The previous formulation, derived from tensor calculus, is incomplete because it is not coordinate covariant. It is written in a particular coordinate system (“special coordinates”) which is not uniquely selected. In fact, eq.(2.1) is left invariant under particular coordinate changes of the $X^A \rightarrow \tilde{X}^A$ with some new function $\tilde{F}(\tilde{X})$ described by

$$\begin{aligned}\tilde{X}^A(X) &= A^A_B X^B + B^{AB} F_B(X) + P^A \\ \tilde{F}_A(\tilde{X}^A(X)) &= C_{AB} X^B + D_A^B F_B(X) + Q_A ,\end{aligned}\tag{2.5}$$

where $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is an $Sp(2r, \mathbb{R})$ matrix

$$A^T C - C^T A = 0 \quad , \quad B^T D - D^T B = 0 \quad , \quad A^T D - C^T B = \mathbb{1} \quad ,\tag{2.6}$$

and P^A, Q_A can be complex constants which from now on will be set to zero.

It can be shown that[3]

$$\tilde{F}_A = \frac{\partial \tilde{F}}{\partial \tilde{X}^A} \quad ,\tag{2.7}$$

provided the mapping $X^A \rightarrow \tilde{X}^A$ is invertible.

It is well known that the equations of motion and the Bianchi identities [3][16][17]

$$\begin{aligned}\partial^\mu \text{Im } \mathcal{F}_{\mu\nu}^{-A} &= 0 && \text{Bianchi identities} \\ \partial_\mu \text{Im } G_{-A}^{\mu\nu} &= 0 && \text{Equations of motion}\end{aligned}\tag{2.8}$$

transform covariantly under (2.5) (with $P^A = Q_A = 0$), so that $(\mathcal{F}_{\mu\nu}^{-A}, G_{-A}^{\mu\nu})$ is a symplectic vector. Here, $G_{-A}^{\mu\nu} \equiv i \frac{\delta \mathcal{L}}{\delta \mathcal{F}_{\mu\nu}^{-A}} = \overline{\mathcal{N}}_{AB} \mathcal{F}_{\mu\nu}^{-B} + \text{fermionic terms}$, where we have set $F_{AB} = \overline{\mathcal{N}}_{AB}$ in order to unify the notations to the gravitational case[3]. The transformations (2.5) leave invariant the whole lagrangian but the vector kinetic term. Indeed, neglecting for the moment fermion terms (see section 2.2) and setting for simplicity $\mathcal{F}_{\mu\nu}^{-A} = \mathcal{F}^A$ and $G_{-A}^{\mu\nu} = G_A$ the vector kinetic lagrangian transforms as follows

$$\begin{aligned}\text{Im } \mathcal{F}^A \overline{\mathcal{N}}_{AB} \mathcal{F}^B &\rightarrow \text{Im } \tilde{\mathcal{F}}^A \tilde{G}_A = \\ &= \text{Im } (\mathcal{F}^A G_A + 2\mathcal{F}^A (C^T B)_A^B G_B + \\ &+ \mathcal{F}^A (C^T A)_{AB} \mathcal{F}^B + G_A (D^T B)^{AB} G_B) .\end{aligned}\tag{2.9}$$

If $C = B = 0$ the lagrangian is invariant. If $C \neq 0, B = 0$ it is invariant up to a four-divergence. In presence of a topologically non-trivial $\mathcal{F}_{\mu\nu}^{-A}$ background,

$(C^T A)_{AB} \int \text{Im } \mathcal{F}_{\mu\nu}^{-A} \mathcal{F}_{\mu\nu}^{-B} \neq 0$, one sees that in the quantum theory duality transformations must be integral valued in $Sp(2r, \mathbb{Z})[1]$ and transformations with $B = 0$ will be called perturbative duality transformations.

If $B \neq 0$ the lagrangian is not invariant. As it is well known, then the duality transformation is only a symmetry of the equations of motion and not of the lagrangian.

Since $\tilde{G}_{-A}^{\mu\nu} = \tilde{\mathcal{N}}_{AB} \tilde{\mathcal{F}}_{\mu\nu}^{-B}$ one also has

$$\tilde{\mathcal{N}} = (C + D\mathcal{N})(A + B\mathcal{N})^{-1} . \quad (2.10)$$

Note that $B \neq 0$ means that the coupling constant $\tilde{\mathcal{N}}$ is inverted and transformations with $B \neq 0$ will be called quantum non perturbative duality symmetries.

The perturbative duality rotations are of the form

$$\begin{pmatrix} A & 0 \\ C & (A^T)^{-1} \end{pmatrix} , \quad A \in GL(r) , \quad A^T C \text{ symmetric} . \quad (2.11)$$

In rigid supersymmetry the tree level symmetries are of the form $\begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix}$ while the quantum perturbative monodromy introduces a $C \neq 0$.

If the original unbroken gauge group is $G = SU(r+1)$, then $A \in$ Weyl group and $A^T C$ is the Cartan matrix $\langle \alpha_i | \alpha_j \rangle$ of $SU(r+1)[2]$.

Special coordinates do not give a coordinate free description of the effective action. A coordinate free description is obtained by introducing a holomorphic symplectic bundle $V = (X^A(z), F_A(z))$ and holomorphic $(1,0)$ forms on the Kähler manifold[4,1]

$$U_i \equiv \partial_i V = (\partial_i X^A, \partial_i F_A) \quad \text{with } i = 1, \dots, r . \quad (2.12)$$

In rigid special geometry the U_i satisfy the constraints[4]

$$\begin{aligned} \mathcal{D}_i U_j &= i C_{ijk} g^{k\bar{l}} \bar{U}_{\bar{l}} \\ \partial_i \bar{U}_{\bar{j}} &= 0 . \end{aligned} \quad (2.13)$$

From these one may derive the metric and the tensor C_{ijk}

$$\begin{aligned} g_{i\bar{j}} &= \partial_i \partial_{\bar{j}} K = i(\partial_{\bar{j}} \bar{F}_A \partial_i X^A - \partial_{\bar{j}} \bar{X}^A \partial_i F_A) \\ &= i \partial_i X^A \partial_{\bar{j}} \bar{X}^B (\mathcal{N}_{AB} - \bar{\mathcal{N}}_{AB}) , \end{aligned} \quad (2.14)$$

$$\begin{aligned}
C_{ikp} &= \partial_i X^A \mathcal{D}_k \partial_p F_A - \partial_i F_A \mathcal{D}_k \partial_p X^A \\
&= \partial_i X^B (\partial_k \partial_p F_B - \partial_k \partial_p X^A \overline{\mathcal{N}}_{AB}) ,
\end{aligned} \tag{2.15}$$

where we have used

$$\partial_{\overline{\tau}} \overline{F}_A = \mathcal{N}_{AB} \partial_{\overline{\tau}} \overline{X}^B . \tag{2.16}$$

The integrability conditions on (2.13) yields

$$R_{i\overline{j}k\overline{l}} = -C_{ikp} \overline{C}_{j\overline{l}p} g^{p\overline{p}} . \tag{2.17}$$

The Bianchi identities of (2.17) also imply that C_{ijk} is a holomorphic completely symmetric tensor obeying $\mathcal{D}_{[i} C_{j]kl} = 0$.

Note that from (2.15) it also follows

$$C_{ijk} = \partial_i X^A \partial_j X^B \partial_k X^C \partial_A \partial_B \partial_C F , \tag{2.18}$$

which in special coordinates reduces to

$$C_{ABC} = \partial_A \partial_B \partial_C F . \tag{2.19}$$

2.2 Fermions

The coordinate free description of fermions is given by $SU(2)$ doublets $(\lambda^{iI}, \lambda_{\overline{I}}^{\overline{i}})$ where upper and lower $SU(2)$ indices I mean positive and negative chiralities respectively[3][27][24]. As such the spinors are symplectic invariant and contravariant vector fields. The antiselfdual field strength $\mathcal{F}_{\alpha\beta}^{-A}$ and positive chiralities spinors are in the same $N = 2$ multiplet, which is, in two component spinor notation,[★]

$$(X^A, \partial_i X^A \lambda_{\alpha}^{iI}, \mathcal{F}_{\alpha\beta}^{-A}) , \tag{2.20}$$

with $\alpha, \beta \in SL(2, \mathbb{C})$.

It is easy to see that a Pauli term which is both coordinate and symplectic invariant up to four fermion terms is uniquely fixed to be[27]

$$\mathcal{L}_{\text{Pauli}}(\lambda) = a \text{ Im } [(\partial_{\overline{\tau}} \overline{X}^A G_{-A}^{b, \alpha\beta} - \partial_{\overline{\tau}} \overline{F}_A \mathcal{F}^{-A\alpha\beta}) g^{\overline{i}j} C_{jkp} \lambda_{\alpha}^{kI} \lambda_{\beta}^{pJ} \epsilon_{IJ}] , \tag{2.21}$$

★ $\mathcal{F}_{\alpha\beta}^{-A}$ is $\sigma_{\alpha\beta}^{\mu\nu} \mathcal{F}_{\mu\nu}^{-A}$.

where $G_{-A}^b \equiv \frac{i\partial\mathcal{L}_{\text{gauge}}}{\partial\mathcal{F}^{-A}}$ and a is a real constant fixed by supersymmetry. The full field strength then contains fermionic corrections, as due to (2.21)

$$G_{-A}^{\alpha\beta} \equiv \frac{i\partial\mathcal{L}}{\partial\mathcal{F}_{\alpha\beta}^{-A}} = \bar{\mathcal{N}}_{AB}\mathcal{F}^{-B\alpha\beta} + \frac{1}{2}a\partial_{\bar{t}}\bar{X}^B(\bar{\mathcal{N}}_{BA} - \mathcal{N}_{BA})g^{\bar{t}j}C_{j\bar{k}p}\lambda^{\alpha kI}\lambda^{\beta pJ}\epsilon_{IJ} . \quad (2.22)$$

Replacing G_{-A}^b by G_{-A} in $\mathcal{L}_{\text{Pauli}}$ does not yet lead to a full duality invariant action.

According to a general argument given in ref [16], duality of the full action (2.4) determines those four fermion terms which are not invariant by themselves. In fact, it is sufficient to demand that (2.4) be duality invariant at the point $\frac{\partial\mathcal{L}}{\partial\mathcal{A}^A} = 0$ where \mathcal{A}^A is the gauge potential. Dropping $\mathcal{L}_{\text{scalar}}$ and $\mathcal{L}_{\text{fermions}}$ (the first line of (2.4)), which are already invariant, we consider only $\hat{\mathcal{L}} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{Pauli}} + \mathcal{L}_{4\text{-fermi}}$. One finds

$$\hat{\mathcal{L}}(\text{Im } \partial^\mu G_{\mu\nu A}^- = 0) = \frac{1}{2}\mathcal{L}_{\text{Pauli}} + \mathcal{L}_{4\text{-fermi}} \quad (2.23)$$

and the coefficient of the non-invariant 4-fermi term is fixed by replacing here G_{-A}^b with G_{-A} as given by (2.22). This yields

$$\begin{aligned} \mathcal{L}_{4\text{-fermi}} = & \frac{a^2}{4}\partial_{\bar{t}}\bar{X}^A\partial_{\bar{l}}\bar{X}^B(\bar{\mathcal{N}}_{AB} - \mathcal{N}_{AB})g^{\bar{t}j}g^{\bar{l}q}C_{j\bar{k}p}C_{qrs}\lambda^{k\alpha I}\lambda^{p\beta J}\epsilon_{IJ}\lambda_{\alpha}^{rK}\lambda_{\beta}^{sL}\epsilon_{KL} + \text{h.c.} \\ & + \text{invariant terms} , \end{aligned} \quad (2.24)$$

in agreement with Cremmer et al. [31].

In special coordinates, setting $\lambda_{\alpha}^{i1} = \chi_{\alpha}^i$, $\lambda_{\alpha}^{i2} = \lambda_{\alpha}^i$ and using eq.(2.21), the Pauli term reduces to

$$\mathcal{L}_{\text{Pauli}} = a \text{Re } \partial_A\partial_B\partial_C F(\chi_{\alpha}^A\lambda_{\beta}^B - \lambda_{\alpha}^A\chi_{\beta}^B)\mathcal{F}^{-C\alpha\beta} , \quad (2.25)$$

in agreement with the standard $N = 1$ supersymmetric action with $f_{AB} = F_{AB}$ [31]. We see from (2.21) and (2.25) that in rigid supersymmetry the physical meaning of C_{ijk} is that of an anomalous magnetic moment. Note that C_{ijk} vanishes at tree-level and it is $\sim \frac{1}{\langle X \rangle}$ at one loop-level as it must be [1,18]. It is obviously singular at $\langle X \rangle = 0$. In the $SU(2)$ theory, because of non-perturbative corrections[1], one expects such terms to behave instead as $\frac{c_0}{\Lambda}$ where c_0 is a dimensionless number. The vanishing at tree-level of both Pauli terms and the corresponding four fermions terms is consistent with renormalizability arguments.

The other fermionic terms which are already duality invariant read

$$\lambda_\alpha^{iI} \lambda_\beta^{kJ} \epsilon^{\alpha\beta} \bar{\lambda}_{\dot{\alpha}I}^{\bar{j}} \bar{\lambda}_{\dot{\beta}J}^{\bar{l}} \epsilon^{\dot{\alpha}\dot{\beta}} R_{i\bar{j}k\bar{l}} \quad (2.26)$$

and

$$\mathcal{D}_i C_{jlm} \lambda_\alpha^{iI} \lambda_\beta^{jK} \epsilon^{\alpha\beta} \lambda_\gamma^{lJ} \lambda_\delta^{mL} \epsilon^{\gamma\delta} \epsilon_{IJ} \epsilon_{KL} . \quad (2.27)$$

Note that, because of eq. (2.17), all couplings in the lagrangian are expressed through the tensors C_{ijk} .

From a tensor calculus point of view all quartic terms but the last come from the equations of motion of the Y_{IJ}^i auxiliary field triplet[3].

2.3 Positivity and monodromies

Let us consider a submanifold \mathcal{M}_r of the moduli space of a Riemann surface of genus r such that its tangent space is isomorphic to the Hodge bundle. In particular the dimension of \mathcal{M}_r is equal to the genus r of the Riemann surface \mathcal{C}_r [★]. In this case, decomposing an abelian differential in terms of the $2r$ harmonic forms dual to the canonical basis of cycles, we have

$$\begin{aligned} \omega &= X^A(z^i) \alpha_A + F_A(z^i) \beta^A \quad A, i = 1, \dots, r \\ \int \alpha_A \wedge \beta^B &= \delta_A^B \quad , \quad \int \alpha_A \wedge \alpha_B = \int \beta_A \wedge \beta_B = 0 , \end{aligned} \quad (2.28)$$

where z^i are coordinates on the moduli space submanifold, and

$$\partial_i \omega = \partial_i X^A \alpha_A + \partial_i F_A \beta^A . \quad (2.29)$$

Then the metric, given by the norm

$$g_{i\bar{j}} = i \int \partial_i \omega \wedge \partial_{\bar{j}} \bar{\omega} = i \partial_i \partial_{\bar{j}} \int \omega \wedge \bar{\omega} \quad (2.30)$$

is manifestly positive. Using eqs. (2.28), (2.29) we find

★ We are aware of the fact that to find an intrinsic characterization of such an algebraic locus is far from obvious. We thank D. Dubrovin, P. Fré and Reina for clarifying discussions on this point.

$$\begin{aligned}
g_{i\bar{j}} &= i\partial_i\partial_{\bar{j}}(\bar{F}_A X^A - \bar{X}^A F_A) = \\
&= i(\partial_i X^A \partial_{\bar{j}} \bar{F}_A - \partial_i F^A \partial_{\bar{j}} \bar{X}^A) = \\
&= i\partial_i X^A \partial_{\bar{j}} \bar{X}^B (\mathcal{N}_{AB} - \bar{\mathcal{N}}_{AB}) ,
\end{aligned} \tag{2.31}$$

which coincides with the metric of $N = 2$ rigid special geometry [1,4].

Formula (2.29) implies by supersymmetry a similar expansion for the full multiplet (2.20). For the upper component $\mathcal{F}_{\mu\nu}^{-A}$ we get a self dual three form

$$w = \mathcal{F}^A \alpha_A + G_A \beta^A \tag{2.32}$$

on $\mathbf{R}_4 \times \mathcal{C}_r$ when (2.8) hold. We observe that in six dimensions an abelian vector multiplet is dual to a tensor multiplet containing a self-dual three form. This remarkable coincidence actually suggests a physical picture for the characterization of this subclass \mathcal{C}_r of Riemann surfaces. Namely, they should appear in the compactification on $\mathbf{R}_4 \times \mathcal{C}_r$ of $N = 1$ six-dimensional theory of a self interacting tensor multiplet.

As shown in ref. [4], the Picard–Fuchs equations for \mathcal{C}_r have a general form dictated by the differential constraints of rigid special geometry. A general proposal for \mathcal{C}_r has been given in [2] and can be used to write down the Picard–Fuchs equations for the periods and to determine their monodromies. Such proposal can be checked by comparing the explicit form of the Picard–Fuchs equations with their general form given by rigid special geometry.

In the one parameter case ($G = SU(2)$), where \mathcal{C}_1 is given by the elliptic curve of ref. [1], the special geometry equations reduce to one ordinary second order equation

$$\left(\frac{d}{dz} + \hat{\Gamma}\right)C^{-1}\left(\frac{d}{dz} - \hat{\Gamma}\right)U = 0 \tag{2.33}$$

where $\hat{\Gamma} = \frac{d}{dz} \log e$, $e = \frac{dX}{dz}$ and C is the 3-tensor appearing in (2.13). This agrees with the Picard–Fuchs equations derived from \mathcal{C}_1 . The general solution of this equation is[4]

$$U = \left(e, e \frac{d^2 F}{dX^2}\right) , \tag{2.34}$$

with $\tau = \frac{\partial^2 F}{\partial X^2}$ being the uniformizing variable for which the differential equation reduces to $\frac{d^2}{d\tau^2}(\) = 0$.

3 Coupling to gravity

The coupling to gravity modifies the constraints of rigid special geometry because of the introduction of a $U(1)$ connection due to the $U(1)$ Kähler –Hodge structure of moduli space. For n vector multiplets one introduces $2(n+1)$ covariantly holomorphic sections [24–26,23,27]

$$V = (L^\Lambda, M_\Lambda) \quad (\Lambda = 0, \dots, n) , \quad (3.1)$$

where 0 is the graviphoton index.

The new differential constraints of special geometry are

$$\begin{aligned} U_i &\equiv (\mathcal{D}_i L^\Lambda, \mathcal{D}_i M_\Lambda) = (f_i^\Lambda, h_{i\Lambda}) \\ \mathcal{D}_i U_j &= i C_{ijk} g^{k\bar{l}} \bar{U}_{\bar{l}} \\ \mathcal{D}_i \bar{U}_{\bar{j}} &= g_{i\bar{j}} \bar{V} \\ \mathcal{D}_i \bar{V} &= 0 , \end{aligned} \quad (3.2)$$

where now \mathcal{D}_i is the covariant derivative with respect to the usual Levi-Civita connection and the Kähler connection $\partial_i K$. That is, under $K \rightarrow K + f + \bar{f}$ a generic field ψ^i which under $U(1)$ transforms as $\psi^i \rightarrow e^{-(\frac{p}{2}f + \frac{\bar{p}}{2}\bar{f})} \psi^i$ has the following covariant derivative

$$\mathcal{D}_i \psi^j = \partial_i \psi^j + \Gamma_{ik}^j \psi^k + \frac{p}{2} \partial_i K \psi^j , \quad (3.3)$$

and analogously for $\mathcal{D}_{\bar{i}}$ with $p \rightarrow \bar{p}$. The (L^Λ, M_Λ) have been given conventionally weights $p = -\bar{p} = 1$.

Since L^Λ, M_Λ are covariantly holomorphic, it is convenient to introduce holomorphic sections $X^\Lambda = e^{-K/2} L^\Lambda$, $F_\Lambda = e^{-K/2} M_\Lambda$.

The Kähler potential is fixed by the condition[3][24]

$$i(\bar{L}^\Lambda M_\Lambda - L^\Lambda \bar{M}_\Lambda) = 1 \quad (3.4)$$

to be

$$K = -\log i(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda) . \quad (3.5)$$

As it is well known[3][32], the differential constraints (3.2) can in general be solved in terms of a holomorphic function homogeneous of degree two $F(X)$. However, as we will see in the sequel, there exist particular symplectic sections for which such prepotential F does not exist. In particular this is the case appearing in the effective

theory of the $N = 2$ heterotic string. For this reason it is convenient to have the fundamental formulas of special geometry written in a way independent of the existence of F .

First of all we note that quite generally we may write

$$h_{i\Lambda} = \overline{\mathcal{N}}_{\Lambda\Sigma} f_i^\Sigma . \quad (3.6)$$

Indeed, when F exists, $\mathcal{N}_{\Lambda\Sigma}$ has the form[3][27]

$$\mathcal{N}_{\Lambda\Sigma} = \overline{F}_{\Lambda\Sigma} + 2i \frac{(\text{Im } F_{\Lambda\Gamma})(\text{Im } F_{\Sigma\Pi}) L^\Gamma L^\Pi}{(\text{Im } F_{\Xi\Omega}) L^\Xi L^\Omega} , \quad (3.7)$$

which turns out to be the coupling matrix appearing in the kinetic term of the vector fields. However, as we show below, (3.6) is symplectic covariant and therefore it always holds even in some specific coordinate system in which F does not exist.

In the same way as in the rigid case, from eqs. (3.2) we find

$$g_{i\bar{j}} = i(f_i^\Lambda \overline{h_{\bar{j}\Lambda}} - h_{i\Lambda} \overline{f_{\bar{j}}^\Lambda}) = i(\mathcal{N}_{\Lambda\Sigma} - \overline{\mathcal{N}}_{\Lambda\Sigma}) f_i^\Lambda \overline{f_{\bar{j}}^\Sigma} \quad (3.8)$$

$$C_{ijk} = f_i^\Lambda \mathcal{D}_j h_{k\Lambda} - h_{i\Lambda} \mathcal{D}_j f_k^\Lambda = f_i^\Lambda \partial_j \overline{\mathcal{N}}_{\Lambda\Sigma} f_k^\Sigma , \quad (3.9)$$

which are symplectic invariant. (Note that $\mathcal{N}_{\Lambda\Sigma}$ has zero Kähler weight).

Furthermore, the integrability conditions (3.2) give[3][24][25][26][23][27]

$$R_{i\bar{j}l\bar{k}} = g_{i\bar{j}} g_{l\bar{k}} + g_{i\bar{k}} g_{l\bar{j}} - C_{ilp} C_{\bar{j}\bar{k}\bar{p}} g^{p\bar{p}} , \quad (3.10)$$

replacing eq. (2.6).

Here C_{ilp} is a covariantly holomorphic tensor of weight $p = -\bar{p} = 2$,

$$\mathcal{D}_{\bar{l}} C_{ijk} = \partial_{\bar{l}} C_{ijk} - \partial_{\bar{l}} K C_{ijk} = 0 , \quad (3.11)$$

which implies $\partial_{\bar{l}} W_{ijk} = 0$ with $C_{ijk} = e^K W_{ijk}$.

Duality transformations are now in $Sp(2n+2, \mathbb{Z})$ and act on X^Λ, F_Λ as in the rigid case. The symplectic action on (L^Λ, M_Λ) (or X^Λ, F_Λ) is

$$\begin{pmatrix} L \\ M \end{pmatrix}' = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} L \\ M \end{pmatrix} = \mathcal{S} \begin{pmatrix} L \\ M \end{pmatrix} \quad \mathcal{S} \in Sp(2n+2, \mathbb{Z}) . \quad (3.12)$$

Then it follows, because of eq. (3.2) and (3.6),

$$\begin{pmatrix} f_i^\Lambda \\ h_{i\Lambda} \end{pmatrix}' = \begin{pmatrix} A & B\overline{\mathcal{N}} \\ C & D\overline{\mathcal{N}} \end{pmatrix} \begin{pmatrix} f_i^\Lambda \\ f_i^\Lambda \end{pmatrix} , \quad (3.13)$$

which also implies

$$\tilde{\mathcal{N}} = (C + D\mathcal{N})(A + B\mathcal{N})^{-1} . \quad (3.14)$$

These two transformations laws imply the covariance of (3.6).

The symplectic action on $\mathcal{F}_{\mu\nu}^{+\Lambda}, G_{+\Lambda}^{\mu\nu}$ is the same as on (L^Λ, M_Λ) , so eq. (2.8) is unchanged. Therefore the discussion of the previous section on perturbative and non perturbative duality transformations in the rigid case remains unchanged when gravity is turned on.

When the sections (X^Λ, F_Λ) are chosen in such a way that a function F exists[★], from (3.4) and the degree two homogeneity of F it follows that [24][27]

$$\text{Im } F_{\Lambda\Sigma} L^\Lambda \bar{f}_i^\Sigma = 0 , \quad (3.15)$$

so that eq. (3.6) becomes $h_{i\Lambda} = F_{\Lambda\Sigma} f^{\Sigma i}$. Furthermore from (3.9) and (3.15) it also follows

$$e^{K/2} C_{ijk} = f_i^\Lambda f_j^\Gamma f_k^\Sigma F_{\Lambda\Gamma\Sigma} . \quad (3.16)$$

By the same token, we have

$$\begin{pmatrix} f_i^\Lambda \\ h_{i\Lambda} \end{pmatrix}' = \begin{pmatrix} A & B\mathcal{F} \\ C & D\mathcal{F} \end{pmatrix} \begin{pmatrix} f_i^\Lambda \\ f_i^\Lambda \end{pmatrix} , \quad (3.17)$$

where $\mathcal{F} = F_{\Lambda\Sigma}$. Note that in these cases

$$\begin{aligned} 2\tilde{F}(\tilde{X}) &= \tilde{F}_\Lambda \tilde{X}^\Lambda = \\ &= 2F + 2X^\Lambda (C^T B)_\Lambda^\Sigma F_\Sigma + X^\Lambda (C^T A)_{\Lambda\Sigma} X^\Sigma + F_\Lambda (D^T B)^{\Lambda\Sigma} F_\Sigma . \end{aligned} \quad (3.18)$$

Note also that the homogeneity of F implies

$$\tilde{X} = (A + B\mathcal{F})X , \quad (3.19)$$

★ A resumé of the duality transformations for this case, including the supergravity corrections has been given in appendix C of [32].

where $\mathcal{F} = F_{\Lambda\Sigma}$ and

$$\tilde{F} = (C + D\mathcal{F})X . \quad (3.20)$$

Special coordinates in supergravity are defined by $t^\Lambda = X^\Lambda/X^0$ since we now have a set of $n + 1$ homogeneous coordinates. If we assume that $\mathcal{D}_i(\frac{X^\Lambda}{X^0})$ is an invertible matrix, then we may choose a frame for which $\partial_i(\frac{X^\Lambda}{X^0}) = \delta_i^\Lambda$. This is possible only if X^Λ are unconstrained variables and so $F_\Lambda = F_\Lambda(X)$, which implies $F_\Lambda = \partial_\Lambda F(X)$ with F homogeneous of degree 2.

We now discuss the possible non-existence of $F(X)$. If we start with some special coordinates X^Λ , $F_\Lambda(X)$, it is possible that in the new basis the \tilde{X}^Λ are not good special coordinates in the sense that the mapping $X \rightarrow \tilde{X}$ is not invertible. This happens whenever the $(n + 1) \times (n + 1)$ matrix $A + B\mathcal{F}$ is not invertible (its determinant vanishes). This does not mean that \tilde{X}, \tilde{F} are not good symplectic sections since the symplectic matrix $\mathcal{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is always invertible. It simply means that $\tilde{F}_\Lambda \neq \tilde{F}_\Lambda(\tilde{X})$ and therefore a prepotential $\tilde{F}(\tilde{X})$ does not exist. However our formulation of special geometry never explicitly used the fact that F_Λ be a functional of the X 's and indeed the quantities (X^Λ, F_Λ) , $(f_i^\Lambda, h_{i\Lambda})$, $\mathcal{N}_{\Lambda\Sigma}$ and $C_{ijk}, g_{i\bar{j}}$ are well defined for any choice of the symplectic sections (X^A, F_A) since they are symplectic invariant or covariant. For example, to compute the “gauge coupling” $\tilde{\mathcal{N}}$ in such a basis $(\tilde{X}^\Lambda, \tilde{F}_\Lambda)$ one uses the formula

$$\tilde{\mathcal{N}}(\tilde{X}, \tilde{F}) = (C + D\mathcal{N}(X))(A + B\mathcal{N}(X))^{-1} , \quad (3.21)$$

and expresses the $X = X(\tilde{X}, \tilde{F})$ by using the fact that the symplectic mapping can be inverted. All other quantities can be computed in this way.

We will see the relevance of this observation in the sequel, while discussing low energy effective action of $N = 2$ heterotic string. A simple example is the following. Consider $F = iX^0X^1$, leading to

$$\mathcal{N} = \begin{pmatrix} i\frac{X^1}{X^0} & 0 \\ 0 & i\frac{X^0}{X^1} \end{pmatrix} . \quad (3.22)$$

This appears in the $N = 2$ reduction of pure $N = 4$ supergravity in the so-called $SO(4)$ formulation [33]. Consider now the symplectic mapping defined by

$$A = D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} ; \quad C = -B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} . \quad (3.23)$$

Then the transformation is

$$\begin{aligned}\tilde{X}^0 &= X^0 & \tilde{X}^1 &= -F_1 \\ \tilde{F}_0 &= F_0 & \tilde{F}_1 &= X^1 .\end{aligned}\tag{3.24}$$

Using in the first line $F_1 = iX^0$ would lead to a non-invertible mapping $X \rightarrow \tilde{X}$, and using (3.18) would lead to $\tilde{F} = 0$. One observes also that $A + B\mathcal{F}$ is non-invertible. However, $A + B\mathcal{N}$ is invertible, and one obtains $\tilde{N} = iX^1(X^0)^{-1}\mathbb{1} = i\tilde{F}_1(\tilde{X}^0)^{-1}\mathbb{1}$. This form appears in the $N = 2$ reduction of the $SU(4)$ formulation of pure $N = 4$ supergravity [34]. These two forms of the $N = 2$ reduced action and the duality transformation have been studied in [35] to relate electric and magnetic charges of black holes.

As far as the fermions are concerned, the gravitational $N = 2$ multiplet is now

$$(L^\Lambda, f_i^\Lambda \lambda^{iI}, \mathcal{F}_{\alpha\beta}^{-\Lambda}) ,\tag{3.25}$$

and, correspondingly, the gaugino Pauli terms have the form

$$a \operatorname{Im} [(\mathcal{D}_{\bar{t}} \bar{L}^\Lambda G_{-\Lambda}^{\alpha\beta} - \mathcal{D}_{\bar{t}} \bar{M}_\Lambda \mathcal{F}^{-\Lambda\alpha\beta}) g^{\bar{t}j} C_{jkl} \lambda_\alpha^{kI} \lambda_\beta^{lJ} \epsilon_{IJ}] ,\tag{3.26}$$

quite analogous to eq. (2.21).

The Pauli terms for gravitino currents which are manifestly duality invariant are easy to find. They are [3][31][27]

$$\begin{aligned}& a_1 \operatorname{Im} [(\mathcal{F}_{\mu\nu}^{-\Lambda} M_\Lambda - G_{\mu\nu-\Lambda} L^\Lambda) \bar{\psi}_\rho^I \psi_\sigma^J \epsilon_{IJ} \epsilon^{\mu\nu\rho\sigma}] + \\ & + a_2 \operatorname{Im} [(\mathcal{F}_{\mu\nu}^{-\Lambda} \mathcal{D}_{\bar{t}} \bar{M}_\Lambda - G_{\mu\nu-\Lambda} \mathcal{D}_{\bar{t}} \bar{L}^\Lambda) \bar{\lambda}_I^{\bar{t}} \gamma_\rho \psi_{\sigma J} \epsilon^{IJ} \epsilon^{\mu\nu\rho\sigma}] .\end{aligned}\tag{3.27}$$

Again, as before, they generate unique quartic terms by requiring duality invariance of the action, on the equations of motion of the vector fields. Of course many of these terms are absent in $N = 1$ [31] theories because of the absence of the second gravitino. This is one of the differences between rigid supersymmetry and local supersymmetry. What happens is that in $N = 2$ supergravity, one introduces an extra $(\frac{3}{2}, 1)$ multiplet, with respect to the $N = 1$ case. This has the effect of having extra auxiliary fields in the supergravity multiplet [36]

$$\mathcal{V}_{J\mu}^I , \quad A_\mu , \quad T_{\mu\nu}^- , \quad D\tag{3.28}$$

other than the matter auxiliary field of the vector multiplet Y^{IJ} (traceless, real, symmetric in IJ), $i, j = 1, 2$, *i.e.* a real $SU(2)$ triplet. The meaning of the auxiliary fields is straightforward. The Y 's correspond to the three auxiliary fields of a $N = 1$ vector multiplet and a chiral multiplet. The D auxiliary field gives the equation (3.4) (i.e (3.5)), $T_{\mu\nu}^-$ is the graviphoton (symplectic invariant) combination of the gauge fields $T_{\mu\nu}^- = L^\Lambda G_{-\Lambda}^{\mu\nu} - M_\Lambda \mathcal{F}_{\mu\nu}^{-\Lambda}$, and $\mathcal{V}_{J\mu}^I$, A_μ are the composite $SU(2)$ and $U(1)$ connections of the quaternionic manifold and Kähler–Hodge manifold respectively. Note that comparison between $N = 1$ and $N = 2$ theories shows that the spinors χ^i of the scalar multiplet and λ^Σ of the vector multiplet of the $N = 1$ theory are related to the doublet λ^{iI} of the $N = 2$ theory by

$$\chi^i = \lambda^{i1} \quad , \lambda^\Sigma = f_i^\Sigma \lambda^{i2} \quad (3.29)$$

3.1 The three–form cohomology

We recall that special geometry in $N = 2$ supergravity, unlike rigid special geometry, is suitable for three–form cohomology for Calabi–Yau manifolds. Let's define a holomorphic three–form [25,26]

$$\Omega = X^\Lambda \alpha_\Lambda + F_\Lambda \beta^\Lambda \quad (3.30)$$

where $\alpha_\Lambda, \beta^\Lambda$ is a $2n + 2$ dimensional cohomology basis dual to the $2n + 2$ homology cycles ($n = h_{21}$). Ω is a holomorphic section of a line bundle. Then it follows that if one defines

$$e^{-K} = i \int \Omega \wedge \overline{\Omega} > 0 \quad (3.31)$$

then

$$g_{i\bar{j}} = \frac{-i \int \mathcal{D}_i \Omega \wedge \mathcal{D}_{\bar{j}} \overline{\Omega}}{i \int \Omega \wedge \overline{\Omega}} = -\partial_i \partial_{\bar{j}} \log i \int \Omega \wedge \overline{\Omega} > 0 . \quad (3.32)$$

The $(2n + 2)$ three–forms $\mathcal{D}_i \Omega, \mathcal{D}_{\bar{i}} \overline{\Omega}, \Omega, \overline{\Omega}$ with the cohomology basis (α_A, β^A) correspond to the decomposition

$$H^3(\mathbf{R}) = H^{(2,1)}(\mathbb{C}) + H^{(1,2)}(\mathbb{C}) + H^{(3,0)}(\mathbb{C}) + H^{(0,3)}(\mathbb{C}) . \quad (3.33)$$

Note that since $\Omega = (X^\Lambda, F_\Lambda)$, then $\mathcal{D}_i \Omega = (\mathcal{D}_i X^\Lambda, \mathcal{D}_i F_\Lambda)$, with $f_i^\Lambda = e^{\frac{K}{2}} \mathcal{D}_i X^\Lambda$, $h_{i\Lambda} = e^{\frac{K}{2}} \mathcal{D}_i F_\Lambda$. The relations

$$\int \Omega \wedge \Omega = \int \Omega \wedge \mathcal{D}_i \overline{\Omega} = 0 \quad (3.34)$$

are obvious since $\mathcal{D}_i\Omega = \partial_i\Omega - \frac{1}{(\Omega, \overline{\Omega})}(\partial_i\Omega, \overline{\Omega})\Omega$. However the relation

$$\int \Omega \wedge \mathcal{D}_i\Omega = 0 , \quad (3.35)$$

which is suitable for three-form cohomology, implies

$$\int \Omega \wedge \partial_i\Omega = 0 , \quad (3.36)$$

i.e.

$$\partial_i X^\Lambda F_\Lambda - \partial_i F_\Lambda X^\Lambda = 0 \quad (3.37)$$

for any choice of the symplectic section. Eq. (3.37) is equivalent to

$$X^\Lambda \mathcal{D}_i F_\Lambda - \mathcal{D}_i X^\Lambda F_\Lambda = 0 . \quad (3.38)$$

From $\mathcal{D}_i F_\Lambda = \overline{\mathcal{N}}_{\Lambda\Sigma} \mathcal{D}_i X^\Sigma$, applying $\mathcal{D}_{\overline{j}}$ to both sides we also find

$$\mathcal{D}_{\overline{j}} \mathcal{D}_i F_\Lambda = \partial_{\overline{j}} \overline{\mathcal{N}}_{\Lambda\Sigma} \mathcal{D}_i X^\Sigma + \overline{\mathcal{N}}_{\Lambda\Sigma} \mathcal{D}_{\overline{j}} \mathcal{D}_i X^\Sigma , \quad (3.39)$$

which implies, using the third line of (3.2),

$$F_\Lambda = \overline{\mathcal{N}}_{\Lambda\Sigma} X^\Sigma + \frac{1}{n} g^{i\overline{j}} \partial_{\overline{j}} \overline{\mathcal{N}}_{\Lambda\Sigma} \mathcal{D}_i X^\Sigma . \quad (3.40)$$

4 Duality symmetries

Duality transformations in generic $N = 2$ supergravity theories are a different choice of the symplectic representative (X^Λ, F_Λ) of the underlying special geometry. If the fields $\mathcal{F}_{\mu\nu}^{+\Lambda}, G_{\mu\nu}^{+\Lambda}$ have no electric or magnetic sources these dualities are simply a different equivalent choice of sections (X^Λ, F_Λ) since they are defined up to a symplectic transformation[3][17]. However if the gauge fields are coupled to (abelian) sources then duality transformations map theories into different theories with a duality transformed source. Since the matrix $\mathcal{N}_{\Lambda\Sigma}$ plays the role of a coupling constant it is clear that in perturbation theory the only possible duality transformations are those with $B = 0$ and of a lower triangular block form

$$\mathcal{S} = \begin{pmatrix} A & 0 \\ C & A^{T-1} \end{pmatrix} . \quad (4.1)$$

Under such change, the action changes in a total derivative which, up to fermion terms, is

$$\mathcal{L}'(A, C) = \mathcal{L} + \text{Im } \mathcal{F}^{-\Lambda} (CA)_{\Lambda\Sigma} \mathcal{F}^{-\Sigma} . \quad (4.2)$$

So the lagrangian is invariant up to a surface term. A duality transformation is a symmetry if

$$\tilde{\mathcal{N}}(\tilde{X}, \tilde{F}) = \mathcal{N}(\tilde{X}, \tilde{F}) = (C + D\mathcal{N}(X, F))(A + B\mathcal{N}(X, F))^{-1} . \quad (4.3)$$

If $F_\Lambda = F_\Lambda(X)$ this implies

$$\tilde{F}(\tilde{X}) = F(\tilde{X}) . \quad (4.4)$$

Then using (3.18) we should have [3][37]

$$\begin{aligned} 2F[(A + B\mathcal{F})X] &= 2F + 2X^\Lambda (C^T B)_\Lambda{}^\Sigma F_\Sigma \\ &\quad + X^\Lambda (C^T A)_{\Lambda\Sigma} X^\Sigma + F_\Lambda (D^T B)^{\Lambda\Sigma} F_\Sigma , \end{aligned} \quad (4.5)$$

which is a functional relation for F given A, B, C, D . Note that because of (3.18) it may happen that $\tilde{F}(\tilde{X}) = 0$. This is so when $\frac{\partial X^\Lambda}{\partial \tilde{X}^\Sigma}$ is not an invertible matrix.

4.1 Heterotic $N = 2$ superstring theories

In $N = 2$ heterotic string theories, as the one obtained by the fermionic construction or by compactification on $T_2 \times K_3$, one often encounters classical moduli spaces which are locally of the form [38][39][18][40][9][41]

$$\frac{O(2, n_v)}{O(2) \times O(n_v)} \times \frac{O(4, n_h)}{O(4) \times O(n_h)} , \quad (4.6)$$

where n_v and n_h are respectively the number of the moduli in vector and hypermultiplets. If there are no charged massless hypermultiplets with respect to the gauge group $U(1)^r$, with $r = n_v$, we may avoid holomorphic anomalies [42][43][44][45] and the situation for this theory may be similar to the rigid Yang–Mills theory coupled to supergravity with an additional dilaton axion multiplet. According to the previous discussion, all perturbative duality symmetries are those for which the previous formula holds for a subgroup of lower triangular matrices

$$\begin{pmatrix} A & 0 \\ C & A^{T-1} \end{pmatrix} \quad (4.7)$$

with $A^T C$ symmetric.

The $(r+2) \times (r+2)$ block A contains the target space T duality and C contains the Peccei–Quinn axion symmetry [11] (for the definition of S in the $N = 2$ context, see below)

$$S \rightarrow S + 1 . \quad (4.8)$$

These are the tree level stringy symmetries of the massive states with $M = |Z|$ where Z is the central charge of the $N = 2$ supersymmetry algebra. If the number of T -moduli is r then the duality symmetries are in $Sp(2r+4; \mathbb{Z})$.

An important point is that we would like to make the tree level (string) symmetry manifest. This means that the gauge fields

$$\mathcal{A}_\mu^A = (G_\mu, B_\mu, \mathcal{A}_\mu^A) \quad A = 2, \dots, r+1 \quad (4.9)$$

(G_μ is the graviphoton and the B_μ is the vector of the dilaton–axion multiplet) should transform in the $2+r$ dimensional (vector) representation of the target space duality symmetry

$$\mathcal{A}' = A\mathcal{A} ; \quad A^T \eta A = \eta ; \quad \eta_{\Lambda\Sigma} = \text{Diag}(1, 1, -1, -1, \dots) , \quad (4.10)$$

with $A \in O(2, r; \mathbb{Z})$. Under the axion Peccei–Quinn symmetry $S \rightarrow S + 1$

$$\mathcal{A}^{\Lambda'} = \mathcal{A}^\Lambda , \quad G_{\Lambda\mu\nu} \rightarrow G_{\Lambda\mu\nu} + \eta_{\Lambda\Sigma} F_{\mu\nu}^\Sigma , \quad (4.11)$$

where

$$\mathcal{N}_{\Lambda\Sigma}(S+1) = \mathcal{N}_{\Lambda\Sigma}(S) + \eta_{\Lambda\Sigma} . \quad (4.12)$$

This formulation is directly obtained by $N = 2$ reduction of the standard form of the $N = 4$ supergravity action[11][7] with a moduli space of the type $O(6, r)/O(6) \times O(r)/\Gamma$ and duality group $\Gamma = O(6; r; \mathbb{Z})$. However to get this in a standard $N = 2$ supergravity form, one must introduce $2+r$ symplectic sections (X^Λ, F_Λ) ($\Lambda = 0, 1, \dots, r+1$) for which $O(2, r)$ is block diagonal and the $S \rightarrow S + 1$ shift is lower triangular. This formulation can be obtained by making a symplectic rotation, with \mathcal{S} given by

$$\mathcal{S} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & -\mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{pmatrix} , \quad (4.13)$$

from a representation in which only $O(2) \times O(r)$ is block diagonal [46], namely

$$\begin{aligned} SO(2, r) : \quad & \begin{pmatrix} A & 0 \\ 0 & \eta A \eta \end{pmatrix} = \mathcal{S} A_1 \mathcal{S}^{-1} \\ S \rightarrow S + 1 : \quad & \begin{pmatrix} \mathbb{1} & 0 \\ \eta & \mathbb{1} \end{pmatrix} = \mathcal{S} A_2 \mathcal{S}^{-1} , \end{aligned} \quad (4.14)$$

where A_1, A_2 are the matrices given in ref. [46]. Since $O(2, r)$ is block diagonal and $S \rightarrow S + 1$ is lower triangular, it follows that the new sections $(\widehat{X}^\Lambda, \widehat{F}_\Lambda)$ are $O(2, r)$ vectors with the property $\widehat{X}^\Lambda \eta_{\Lambda\Sigma} \widehat{X}^\Sigma = \widehat{F}_\Lambda \eta^{\Lambda\Sigma} \widehat{F}_\Sigma = 0$. Under $S \rightarrow S + 1$

$$\begin{aligned} \widehat{X}^\Lambda &\rightarrow \widehat{X}^\Lambda \\ \widehat{F}_\Lambda &\rightarrow \widehat{F}_\Lambda + \eta_{\Lambda\Sigma} \widehat{X}^\Sigma , \end{aligned} \quad (4.15)$$

which induces a Kähler transformation on the Kähler potential. It follows that $\widehat{F}_\Lambda = S \eta_{\Lambda\Sigma} \widehat{X}^\Sigma$ and from eq. (3.18) we find $\widehat{F}(\widehat{X}) = 0$. Note that this is precisely the case for which $\widehat{F}_\Lambda = \widehat{F}_\Lambda(\widehat{X}^\Lambda)$ does not hold. In the same basis the (non-perturbative) inversion $S \rightarrow -\frac{1}{S}$ is given by the symplectic matrix $\begin{pmatrix} 0 & \eta \\ -\eta & 0 \end{pmatrix}$. This element, together with the one corresponding to $S \rightarrow S + 1$ generates an $Sl(2, \mathbb{Z})$ commuting with the $O(2r, \mathbb{Z})$ in $Sp(2r + 4, \mathbb{Z})$. The inversion is actually the only symmetry generator with $B \neq 0$. This transformation will be a symmetry of the spectrum of electrically and magnetically charged states discussed in chapter 5 .

The new sections are given explicitly by eqs. (3.19),(3.20),

$$\begin{aligned} \widehat{X}^\Lambda &= \frac{1}{\sqrt{2}} (\delta_{\Lambda\Sigma} - F_{\Lambda\Sigma}) X^\Sigma \\ \widehat{F}_\Lambda &= \frac{1}{\sqrt{2}} (\delta_{\Lambda\Sigma} + F_{\Lambda\Sigma}) X^\Sigma , \end{aligned} \quad (4.16)$$

where the function

$$F = -\sqrt{X_i^2} \sqrt{X_\alpha^2} \quad i = 0, 1; \quad \alpha = 2, \dots, r + 1 \quad (4.17)$$

was obtained in ref. [46]. The Kähler potential is

$$K = -\log i(\widehat{X}^\Lambda \widehat{F}_\Lambda - \widehat{X}^\Lambda \widehat{F}_\Lambda) = -\log i(\overline{S} - S) - \log \widehat{X}^\Lambda \eta_{\Lambda\Sigma} \widehat{X}^\Sigma . \quad (4.18)$$

The holomorphic sections \widehat{X}^Λ can be written as follows [7]

$$\widehat{X}^\Lambda = \left(\frac{1}{2}(1 + y_\alpha^2), \frac{i}{2}(1 - y_\alpha^2), y^\alpha \right), \quad (4.19)$$

where the y^α are coordinates of the $O(2, r)/O(2) \times O(r)$ manifold. If one introduces the objects

$$\Phi^\Lambda = \frac{\widehat{X}^\Lambda}{\sqrt{\widehat{X}^\Sigma \eta_{\Sigma\Pi} \widehat{X}^\Pi}}, \quad (4.20)$$

then the kinetic matrix $\widehat{\mathcal{N}}_{\Lambda\Sigma}$ is given by [7][9][11]

$$\widehat{\mathcal{N}}_{\Lambda\Sigma}(\widehat{X}) = (S - \overline{S})(\Phi_\Lambda \overline{\Phi}_\Sigma + \overline{\Phi}_\Lambda \Phi_\Sigma) + \overline{S} \eta_{\Lambda\Sigma}, \quad (4.21)$$

where $\Phi_\Lambda = \eta_{\Lambda\Sigma} \Phi^\Sigma$, and we will also further raise or lower indices with η . This can be obtained from the formulae

$$\widehat{\mathcal{N}}(\widehat{X}, \widehat{F}) = (\mathbb{1} + \mathcal{N}(X))(\mathbb{1} - \mathcal{N}(X))^{-1} \quad (4.22)$$

by writing on the right hand side $X^\Lambda = X^\Lambda(\widehat{X}, \widehat{F})$. Formula (4.21) is precisely what is obtained from $N = 4$ supergravity. Because of target space duality we expect that also the $\widehat{X}^\Lambda, \widehat{F}_\Lambda$ become, because of one loop corrections, a lower triangular representation of $Sp(2r + 4, \mathbb{Z})$

$$\begin{pmatrix} \widehat{X}^\Lambda \\ \widehat{F}_\Lambda \end{pmatrix} \rightarrow \begin{pmatrix} A & 0 \\ A^{T-1}C & A^{T-1} \end{pmatrix} \begin{pmatrix} X^\Lambda \\ F_\Lambda \end{pmatrix}, \quad (4.23)$$

where the matrix C comes from the monodromy of the one-loop term [1,2].

5 On monodromies in string effective field theories

We have just seen that the tree-level values of the symplectic sections $(X^\Lambda(z), F_\Lambda(z))$ are given by

$$X^\Lambda \equiv X_{\text{tree}}^\Lambda, \quad F_\Lambda = S \eta_{\Lambda\Sigma} X_{\text{tree}}^\Sigma. \quad (5.1)$$

The target space duality group $O(2, r; \mathbb{Z})$ acts non-trivially on them

$$\Gamma_{\text{cl}} : \begin{pmatrix} X^\Lambda \\ F_\Lambda \end{pmatrix}_{\text{tree}} \rightarrow \begin{pmatrix} A & 0 \\ 0 & \eta A \eta \end{pmatrix} \begin{pmatrix} X^\Lambda \\ F_\Lambda \end{pmatrix}_{\text{tree}}, \quad (5.2)$$

generalizing the action of the Weyl group of the rigid case [2].

At the one loop level, one expects that F_Λ^{tree} is changed to [45]

$$F_\Lambda^{\text{tree}} \rightarrow SX^\Sigma \eta_{\Lambda\Sigma} + f_\Lambda(X) \quad (5.3)$$

where $f_\Lambda(X)$ is a modular covariant structure.

The associated perturbative monodromy can be obtained assuming, according to ref. [1], that the rigid perturbative monodromy does not affect the gravitational sector X^0, X^1, F_0, F_1 . Thus the perturbative lower triangular monodromy matrix is $\Gamma_{\text{cl}}T$, where [1][2]

$$T = \begin{pmatrix} \mathbb{1} & 0 \\ C & \mathbb{1} \end{pmatrix} \quad (5.4)$$

and C is an $(r+2) \times (r+2)$ symmetric matrix with non-vanishing entries on the $r \times r$ block

$$C = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & & \\ \vdots & \vdots & C_{ij} & \\ 0 & 0 & & \end{pmatrix} \quad i, j = 1, \dots, r. \quad (5.5)$$

Indeed, we may think of decomposing $Sp(4+2r)$ into $Sp(4) \times Sp(2r)$ and simply assume that the rigid monodromy $\Gamma_r \in Sp(2r)$ commute with the gravitational $Sp(4)$ sector. This argument should at least apply when the vectors of the Cartan subalgebra of the enhanced gauge symmetry belong to the compact $O(r)$ in $O(2, r)$.

In string theory, the classical stringy moduli space corresponds to the broken phase $U(1)^r$ of several gauge groups with the same rank. For instance, for $r = 2$, $O(2, 2; \mathbb{Z})$ interpolates between $SU(2) \times U(1)$, $SU(2) \times SU(2)$ and $SU(3)$ [47]. In the $N = 4$ theory the $O(6; 22)$ moduli space corresponds to broken phases of several gauge groups of rank 22 such as, $U(1)^6 \times E_8 \times E_8$ or $SO(32) \times U(1)^6$ or $SO(44)$ which are not subgroups one of the other [38].

It is obvious that generically this means that the one loop β -function term [18][19] should have non-trivial monodromies at the points where some higher symmetry is restored. For instance, for $r = 2$ we may expect non trivial monodromies around $t = u$ ($SU(2) \times U(1)$ symmetry restored) and $t = u = i$, $t = u = e^{i\pi/3}$ ($SU(2) \times SU(2)$ or $SU(3)$ symmetry restored), t, u being the parameters defined below.

This means that in supergravity theories derived from strings, because of target space T-duality, the enhanced symmetry points are richer than in the rigid case. Since

different enhancement points are consequence of $O(2, r; \mathbb{Z})$ duality, we expect that a modular invariant treatment of quantum monodromies will automatically ensure non trivial monodromy at the enhanced symmetry points.

In the sequel we shall discuss in some more detail the classical and perturbative monodromies in the $r = 1$ case ($O(2, 1; \mathbb{Z})$) and the classical monodromies for $r = 2$ ($O(2, 2; \mathbb{Z})$).

Consider the tree level prepotential F in the so-called cubic form [3] for $\frac{SU(1,1)}{U(1)} \times \frac{O(2,1)}{O(2)}$:

$$F = \frac{1}{2}(X^0)^2 st^2 , \quad (5.6)$$

where $S = \frac{X^1}{X^0}$ is the dilaton coordinate and $t = \frac{X^2}{X^0}$ is the single modulus of the classical target space duality. We parametrize the $O(2, 1; \mathbb{Z})$ vector as follows

$$\begin{aligned} X^0 &= \frac{1}{2}(1 - t^2) \\ X^1 &= -t \\ X^2 &= -\frac{1}{2}(1 + t^2) \end{aligned} \quad (X^0)^2 + (X^1)^2 - (X^2)^2 = 0 \quad (5.7)$$

The symplectic transformation relating (X^Λ, F_Λ) , $(\Lambda = 0, 1, 2)$ to the $(\hat{X}^\Lambda, \hat{F}_\Lambda)$ where $O(2, 1)$ is linearly realized is easily found to be

$$\begin{pmatrix} \hat{X}^\Lambda \\ \hat{F}_\Lambda \end{pmatrix} = \begin{pmatrix} P & -2R \\ R & 2P \end{pmatrix} \begin{pmatrix} X^\Lambda \\ F_\Lambda \end{pmatrix} , \quad (5.8)$$

where

$$P = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & -1 \\ -\frac{1}{2} & 0 & 0 \end{pmatrix} ; \quad R = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} . \quad (5.9)$$

Let us now implement the t -modulus $Sl(2, \mathbb{Z})$ transformations $t \rightarrow -\frac{1}{t}$, $t \rightarrow t + n$ (note that while $t \rightarrow -\frac{1}{t}$ corresponds to the $SU(2)$ Weyl transformation of the rigid theory, $t \rightarrow t + n$ has no counterpart in the rigid case, being of stringy nature). Using the parametrization (5.7) we find

$$\begin{aligned} t \rightarrow -\frac{1}{t} : \quad & \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \equiv -\eta \in O(2, 1; \mathbb{Z}) \\ t \rightarrow t + n : \quad & \begin{pmatrix} 1 - \frac{n^2}{2} & n & \frac{n^2}{2} \\ -n & 1 & n \\ -\frac{n^2}{2} & n & 1 + \frac{n^2}{2} \end{pmatrix} \equiv V(n) \in O(2, 1; \mathbb{Z}) . \end{aligned} \quad (5.10)$$

Note that (5.10) implies $n \in 2\mathbb{Z}$, i.e. the subgroup $\Gamma_{(0)}(2)$ of $SL(2, \mathbb{Z})$. Actually this gives a projective representation in the subgroup in $O(2, 1; \mathbb{Z})$ of the matrices congruent to the identity *mod* 2.

It follows that Γ_{cl} is generated by (Γ_1, Γ_2) where

$$\begin{aligned}\Gamma_1 &= \begin{pmatrix} -\eta & 0 \\ 0 & -\eta \end{pmatrix} \in Sp(6, \mathbb{Z}) \\ \Gamma_2 &= \begin{pmatrix} V(2) & 0 \\ 0 & \eta V(2)\eta \end{pmatrix} \in Sp(6, \mathbb{Z}) .\end{aligned}\tag{5.11}$$

On the other hand it is possible to go to a stringy basis with a new metric $X_0^2 + X_1^2 - X_2^2 = 2\tilde{X}_1^2 + XY$ such that $SL(2, \mathbb{Z})$ is integral valued in $O(2, 1; \mathbb{Z})$.

The $O(2, 1; \mathbb{Z})$ generators corresponding to translation and inversion are respectively given by:

$$\begin{pmatrix} 1 & -2n & 0 \\ 0 & 1 & 0 \\ n & -n^2 & 1 \end{pmatrix} ; \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} .\tag{5.12}$$

To make contact with the rigid theory it is convenient to define the inversion generator in $O(2, 1; \mathbb{Z})$ with the opposite sign with respect to the previous definition.

Let us now examine the perturbative monodromy matrices T [1]. If we assume as before that the $t \rightarrow -\frac{1}{t}$ pertaining to the rigid theory does not affect the gravitational sector (X^0, X^1, F_0, F_1) , then we have

$$T = \begin{pmatrix} \eta & 0 \\ C & \eta \end{pmatrix} , \quad C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}\tag{5.13}$$

corresponding to the embedding of the $Sp(2, \mathbb{Z})$ rigid transformations acting on the rigid section (X^2, F_2) in $Sp(6, \mathbb{Z})$. Furthermore, considering the transformation of the $\mathcal{N}_{\Lambda\Sigma}$ matrix and setting $D = A = \eta$, $B = 0$ we find

$$\hat{\mathcal{N}}_{22} = -2 + \mathcal{N}_{22}\tag{5.14}$$

for all other entries $\hat{\mathcal{N}}_{\Lambda\Sigma} = \mathcal{N}^{\Lambda\Sigma}$. This is exactly the rigid result[1]. However conjugating the T matrix with Γ_2 one gets

$$C_{\Lambda\Sigma} = \begin{pmatrix} 8 & -8 & -12 \\ -8 & 8 & 12 \\ -12 & 12 & 18 \end{pmatrix}\tag{5.15}$$

which shows that $O(2, 1; \mathbb{Z})$ introduces non-trivial perturbative monodromies for all couplings. The other perturbative lower diagonal monodromy is the dilaton shift (4.14) which commutes with $O(2, 1; \mathbb{Z})$.

Analogous considerations hold for $O(2, n; \mathbb{Z})$, $n > 1$. We limit ourselves to write down the generators of Γ_{cl} for the $O(2, 2; \mathbb{Z})$ case. We use the parametrization of $O(2, 2)/O(2) \times O(2)$ given by

$$\begin{aligned} X^0 &= \frac{1}{2}(1 - tu) \\ X^1 &= -\frac{1}{2}(t + u) \\ X^2 &= -\frac{1}{2}(1 + tu) \\ X^3 &= \frac{1}{2}(t - u) \end{aligned} \quad (X^0)^2 + (X^1)^2 - (X^2)^2 - (X^3)^2 = 0, \quad (5.16)$$

where t, u are the moduli appearing in the F function $F = stu$. In the same way as for the $r = 1$ case it is easy to find the symplectic transformations relating the sections of the cubic parametrization to the X^Λ defined in (5.16). They are given by

$$\begin{pmatrix} X \\ F \end{pmatrix} \rightarrow \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \begin{pmatrix} X \\ F \end{pmatrix}, \quad (5.17)$$

with

$$\begin{aligned} X &= (X^0, X^1, X^2, X^3)^T, \quad F = (F_0, F_1, F_2, F_3)^T \\ A &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad B = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (5.18)$$

It is convenient to use the string basis where the metric η takes the form[11]

$$\eta = \begin{pmatrix} 0 & \mathbb{1}_{2 \times 2} \\ \mathbb{1}_{2 \times 2} & 0 \end{pmatrix}, \quad (5.19)$$

corresponding to the basis $\frac{1}{\sqrt{2}}(X^0 \mp X^2), \frac{1}{\sqrt{2}}(X^1 \mp X^3)$. Then one finds the following

$O(2, 2; \mathbb{Z})$ representation

$$\begin{aligned}
ut &\rightarrow +\frac{1}{ut} : \begin{pmatrix} 0 & -\mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} = \gamma_{ut} \\
t &\rightarrow -\frac{1}{t} : \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix} = \gamma_t \\
u &\rightarrow -\frac{1}{u} : \begin{pmatrix} 0 & \epsilon \\ \epsilon & 0 \end{pmatrix} = \gamma_u \\
t &\rightarrow t + n : \begin{pmatrix} \mathbf{N}^t(-n) & 0 \\ 0 & \mathbf{N}(n) \end{pmatrix} = \gamma_n \\
t &\rightarrow u : \begin{pmatrix} a & b \\ b & a \end{pmatrix} = \gamma ; \quad a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} ; \quad b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned} \tag{5.20}$$

where $\epsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\mathbf{N}(n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$.

Γ_{cl} is then generated by the matrices:

$$\Gamma_{ut} = \begin{pmatrix} \gamma_{ut} & 0 \\ 0 & \gamma_{ut} \end{pmatrix} ; \Gamma_t = \begin{pmatrix} \gamma_t & 0 \\ 0 & \gamma_t \end{pmatrix} ; \Gamma_u = \begin{pmatrix} \gamma_u & 0 \\ 0 & \gamma_u \end{pmatrix} ; \Gamma_n = \begin{pmatrix} \gamma_n & 0 \\ 0 & \gamma_{-n}^T \end{pmatrix} . \tag{5.21}$$

We note that the points $t = u$; $t = u = i$; $t = u = e^{\frac{\pi i}{3}}$ are enhanced symmetry points corresponding to $SU(2) \times U(1)$, $SU(2) \times SU(2)$, and $SU(3)$ respectively [47]. Therefore we expect non-trivial quantum monodromies at these points according to the previous discussion.

5.1 The BPS mass formula

The classical and one loop monodromies are of course reflected in symmetries of the electrically charged massive states belonging to $O(2, n; \mathbb{Z})$ lorentzian lattice[38]. The BPS mass formula [48] in the gravitational case is

$$M = |Z| = |n_{\Lambda}^{(e)} L^{\Lambda} - n_{(m)}^{\Lambda} M_{\Lambda}| = e^{K/2} |n_{\Lambda}^{(e)} X^{\Lambda} - n_{(m)}^{\Lambda} F_{\Lambda}| . \tag{5.22}$$

Note that the central charge Z has definite $U(1)$ weight

$$Z \rightarrow e^{(\bar{f}-f)/2} Z , \tag{5.23}$$

while the mass M is Kähler invariant. The symplectic invariance of M also implies that $(n_{(m)}^\Lambda, n_\Lambda^{(e)})$ transforms as (X^Λ, F_Λ)

$$\begin{pmatrix} n_{(m)}^\Lambda \\ n_\Lambda^{(e)} \end{pmatrix} \rightarrow \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} n_{(m)}^\Lambda \\ n_\Lambda^{(e)} \end{pmatrix}, \quad (5.24)$$

where according to our previous discussion the perturbative symmetries have $B = 0$. Note that $n_{(m)}^\Lambda, n_\Lambda^{(e)}$ must satisfy a lattice condition. We consider here the particular orbit:

$$n_{(m)}^\Lambda n_{(m)}^\Sigma \eta_{\Lambda\Sigma} = n_\Lambda^{(e)} n_\Sigma^{(e)} \eta^{\Lambda\Sigma} = n_{(m)}^\Lambda n_\Lambda^{(e)} = 0. \quad (5.25)$$

In the tree level approximation we may write

$$M = |(n_\Lambda^{(e)} - n_{(m)}^\Sigma \eta_{\Lambda\Sigma} S) X^\Lambda| e^{K/2} \quad (5.26)$$

which is invariant under the tree level symmetry $S \rightarrow S + 1$, but also under the non-perturbative inversion $S \rightarrow -\frac{1}{S}[34][12][11][13][14]$ taking into account that

$$K = -\log i(\bar{S} - S) - \log \frac{X^\Lambda \bar{X}^\Sigma}{M_{Pl}^2} \eta_{\Lambda\Sigma} \quad (5.27)$$

Formula (5.26) is therefore invariant under the $S - T$ duality symmetry $Sl(2, \mathbb{Z}) \times O(2, r; \mathbb{Z}) \subset Sp(2r + 4; \mathbb{Z})$.

The electric mass spectrum can be written as

$$M^2 = |Z|^2 = M_{Pl}^2 \frac{1}{2i(\bar{S} - S)} \mathcal{Q}^{\Lambda\Sigma} n_\Lambda^{(e)} n_\Sigma^{(e)}, \quad (5.28)$$

where $i(\bar{S} - S) = \frac{8\pi}{g^2} > 0$ and $\mathcal{Q}^{\Lambda\Sigma} = \Phi^\Lambda \bar{\Phi}^\Sigma + \bar{\Phi}^\Lambda \Phi^\Sigma$. Formula (5.28) has exactly the same form as the analogous one obtained in $N = 4$ (see ref [11]). When also magnetic charges are present then

$$M^2 = \frac{1}{4} M_{Pl}^2 (n_m, n_e) (\mathcal{M} \mathcal{Q} + \mathcal{L} \hat{\mathcal{Q}}) \begin{pmatrix} n_m \\ n_e \end{pmatrix}, \quad (5.29)$$

where $\mathcal{M} = \frac{1}{\text{Im } S} \begin{pmatrix} S \bar{S} & -\text{Re } S \\ -\text{Re } S & 1 \end{pmatrix}$, $\mathcal{L} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\hat{\mathcal{Q}} = i(\bar{\Phi}^\Lambda \Phi^\Sigma - \Phi^\Lambda \bar{\Phi}^\Sigma)$.

The last term actually vanishes if in solving the constraints (5.25) we set $n_\Lambda^{(e)} = m_1 n_\Lambda$, $n_{(m)}^\Lambda = m_2 n_\Sigma \eta^{\Lambda\Sigma}$, with $n^\Lambda n_\Lambda = 0$.

In the $O(2, 1; \mathbb{Z})$ case the constraint $n_0^2 + n_1^2 - n_2^2 = 0$ is solved by setting

$$\begin{aligned} n_1 &= mr_1 r_2 \\ n_0 - n_2 &= -mr_2^2 \\ n_0 + n_2 &= mr_1^2 , \end{aligned} \tag{5.30}$$

where $m, r_i \in \mathbb{Z}$ and r_1, r_2 are relatively prime integers. If $t \rightarrow t + n$, n even, then if m is odd, r_1, r_2 are odd. Using the parametrization (5.7) we find

$$n_\Lambda X^\Lambda = \frac{M_{Pl}}{2} m (r_1 - r_2 t)^2 , \tag{5.31}$$

and therefore the electric charged states have mass

$$M_{el}^2 = -\frac{M_{Pl}^2}{2} m^2 \frac{(r_1 - r_2 t)^2 (r_1 - r_2 \bar{t})^2}{(t - \bar{t})^2 i(\bar{S} - S)} , \tag{5.32}$$

which at $t = i$ ($X^2 = 0$) ($SU(2)$ restored) becomes

$$M_{el}^2(t = i) = \frac{M_{Pl}^2}{8} \frac{m^2}{i(\bar{S} - S)} (r_1^2 + r_2^2)^2 . \tag{5.33}$$

Similarly, in the $O(2, 2; \mathbb{Z})$ case the constraint $n_0^2 + n_1^2 - n_2^2 - n_3^2 = 0$ is solved by

$$\begin{aligned} n_0 - n_2 &= r_1 r_3 \\ n_1 - n_3 &= r_1 r_4 \\ n_0 + n_2 &= r_2 r_4 \\ n_1 + n_3 &= -r_2 r_3 , \end{aligned} \tag{5.34}$$

where $r_i \in \mathbb{Z}$ and (r_3, r_4) relatively prime. From the parametrizations (5.16) we find

$$n_\Lambda X^\Lambda = \frac{M_{Pl}}{2} (r_2 - r_1 u)(r_4 + r_3 t) , \tag{5.35}$$

and the corresponding electric mass is

$$M_{el}^2 = -\frac{1}{8} M_{Pl}^2 \frac{V^T C(t, \bar{t}) V U^T C(u, \bar{u}) U}{i(\bar{S} - S)(\bar{t} - t)(\bar{u} - u)} , \tag{5.36}$$

where

$$V = \begin{pmatrix} -r_3 \\ r_4 \end{pmatrix} \quad U = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \quad C = \begin{pmatrix} 2\bar{t}t & -(t + \bar{t}) \\ -(t + \bar{t}) & 2 \end{pmatrix} . \tag{5.37}$$

The three enhancement points (5.36) become

$$\begin{aligned}
t = u : \quad M_{el}^2 &= -\frac{1}{8} M_{Pl}^2 \frac{V^T C(t, \bar{t}) V U^T C(t, \bar{t}) U}{i(\bar{S} - S)(\bar{t} - t)^2} \\
t = u = i : \quad &\frac{1}{8} M_{Pl}^2 \frac{V^T V U^T U}{i(\bar{S} - S)} \\
t = u = e^{i\pi/3} : \quad M_{el}^2 &= \frac{1}{24} M_{Pl}^2 \frac{V^T C(e^{i\pi/3}) V U^T C(e^{i\pi/3}) U}{i(\bar{S} - S)} .
\end{aligned} \tag{5.38}$$

The real symmetric matrix $C_{ij}(t, \bar{t})$ is essentially the σ -model metric G_{ij} [10], and it reduces to the Cartan matrix of $SU(2) \times SU(2), SU(3)$, at the enhanced symmetry points $t = i, t = e^{i\pi/3}$. Analogous considerations can be drawn for the classical spectrum of magnetic and dyon charges using the general formula (5.29). Obviously the quantum spectrum is found by substituting $F_{\Lambda\text{tree}} \equiv S\eta_{\Lambda\Sigma} X^\Sigma \rightarrow F_{\Lambda\text{tree}} + \text{quantum corrections}$.

6 Conclusions

In this paper we have formulated electromagnetic duality transformations in generic $D = 4$, $N = 2$ supergravities theories in a form suitable to investigate non-perturbative phenomena. Our formulation is manifestly duality covariant for the full Lagrangian, including fermionic terms, which unlike the rigid case, cannot be retrieved from the $N = 1$ formulation, nor from the $N = 2$ tensor calculus approach. Particular attention has been given to classical T -duality symmetries which actually occur in string compactifications and whose linear action on the gauge potential fields do not allow for the existence of a prepotential function F for the $N = 2$ special geometry. As examples we described the “classical” electric and monopole spectrum for T -duality symmetries of the type $O(2, r; \mathbb{Z})$, with particular details for the $r = 1, 2$ cases, by using the $N = 2$ formalism.

For “classical” monodromies this spectrum is of course related to the spectrum of $N = 4$ theories studied by Sen and Schwarz [11]. Possible extensions of duality symmetries to type II strings have been conjectured by Hull and Townsend [49] and also discussed in [2]. In the present context of $N = 2$ heterotic strings the corresponding type II theories, having $N = 2$ space-time supersymmetry would correspond to $(2, 2)$ superconformal field theories, i.e. quantum Calabi-Yau manifolds.

Due to the non-compact symmetries the BPS saturated states with non-vanishing central charges have a spectrum quite different from the rigid case. Indeed

in rigid theories the “classical” central charge $Z_{(cl)}$ vanishes at the enhanced symmetry points where the original gauge group is restored since there is no dimensional scale other than the Higgs v.e.v.. On the contrary, in the supergravity theory the BPS spectrum at these particular points corresponds in general to electrically and magnetically charged states with Planckian mass (black holes, gravitational monopoles and dyons) [50][11][51][52][53][54]. The only charged states which become massless at the enhanced symmetric point are those with $\eta^{\Lambda\Sigma} n_{\Lambda}^{(e)} n_{\Sigma}^{(e)} < 0$.

We also discussed perturbative monodromies and their possible relations with the rigid case. Non perturbative duality symmetries are more difficult to guess, but it is tempting to conjecture that a quantum monodromy consistent with positivity of the metric and special geometry may be originated by a 3-dimensional Calabi-Yau manifold or its mirror image. If this is the case this manifold should embed in some sense the class of Riemann surfaces studied[1][2] in connection with the moduli space of $N = 2$ rigid supersymmetric Yang-Mills theories.

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References

- [1] N. Seiberg and E. Witten, Nucl. Phys. B426 (1994) 19, hep-th/9407087; Nucl. Phys. B431 (1994) 484, hep-th/9408099.
- [2] A. Klemm, W. Lerche, S. Yankielowicz and S. Theisen, *Simple Singularities and $N = 2$ Supersymmetric Yang-Mills Theory*, preprint CERN-TH.7495/94, LMU-TPW 94/16, hep-th/9411048 and *On the monodromies of $N=2$ supersymmetric Yang-Mills theory*, contribution to these Proceedings, preprint CERN-TH-7538-94, hep-th/9412158; P. C. Argyres and A. E. Faraggi, *The Vacuum Structure and Spectrum of $N = 2$ Supersymmetric $SU(n)$ Gauge Theory*, preprint IASSNS-HEP-94/94, hep-th/9411057
- [3] B. de Wit and A. Van Proeyen, Nucl. Phys. B245 (1984) 89; E. Cremmer, C. Kounnas, A. Van Proeyen, J. P. Derendinger, S. Ferrara, B. De Wit and L. Girardello, Nucl. Phys. B250 (1985) 385; B. de Wit, P. G. Lauwers and A. Van Proeyen, Nucl. Phys. B255 (1985) 569; S. Cecotti, S. Ferrara and L. Girardello, Int. J. Mod. Phys. A4 (1989) 2475.
- [4] A. Ceresole, R. D'Auria and S. Ferrara, Phys. Lett. 339B (1994) 71, hep-th/9408036 .
- [5] D. Amati, K. Konishi, Y. Meurice, G. C. Rossi and G. Veneziano, Phys. Rep. 162 (1988) 169.
- [6] E. Witten, *Monopoles and Four-Manifolds*, preprint IASSNS-HEP-94-96, hep-th/9411102.
- [7] S. Ferrara, L. Girardello, C. Kounnas and M. Porrati, Phys. Lett. B192 (1987) 368.
- [8] N. Seiberg, Nucl. Phys. B303 (1988) 206.
- [9] S. Ferrara and M. Porrati, Phys. Lett. B216 (1989) 289.
- [10] For a review on target space duality see A. Giveon, M. Porrati and E. Rabinovici, Phys. Rep. 244 (1994) 77, hep-th/9401139.
- [11] A. Sen, Nucl. Phys. B388 (1992) 457 and Phys. Lett. 303B (1993) 22; A. Sen and J. H. Schwarz, Nucl. Phys. B411 (1994) 35; Phys. Lett. 312B (1993) 105.
- [12] M. J. Duff and R. R. Khuri, Nucl. Phys. B411 (1994) 473.
- [13] J. Gauntlett and J. A. Harvey, *S-Duality and the Spectrum of Magnetic Monopoles in Heterotic String Theory*, preprint EFI-94-36, hep-th/9407111.

- [14] L. Girardello, A. Giveon, M. Porrati and A. Zaffaroni, Phys. Lett. B334 (1994) 331, hep-th/9406128.
- [15] J. H. Schwarz, *Evidence for Non-perturbative String Symmetries* , preprint CALT-68-1965, hep-th/9411178.
- [16] S. Ferrara, J. Scherk and B. Zumino, Nucl. Phys. B121 (1977) 393.
- [17] M. K. Gaillard and B. Zumino, Nucl. Phys. B193 (1981) 221.
- [18] P. Di Vecchia, R. Musto, F. Nicodemi and R. Pettorino, Nucl. Phys. B252 (1985) 635.
- [19] N. Seiberg, Phys. Lett. 206B (1988) 75.
- [20] S. Ferrara and A. Van Proeyen, Class. Quantum Grav. 6 (1989) 124.
- [21] E. Cremmer and A. Van Proeyen, Class. Quantum Grav. 2 (1985) 445.
- [22] S. Ferrara and J. Louis, Phys. Lett. 278B (1992) 240.
- [23] A. Ceresole, R. D'Auria, S. Ferrara, W. Lerche and J. Louis, Int. J. Mod. Phys. A8 (1993) 79
- [24] L. Castellani, R. D'Auria and S. Ferrara, Phys. Lett. 241B (1990) 57; Class. Quantum Grav. 1 (1990) 317.
- [25] S. Ferrara and A. Strominger, in *Strings '89* , eds. R. Arnowitt, R. Bryan, M.J. Duff, D.V. Nanopoulos and C.N. Pope(World Scientific, 1989), p. 245; A. Strominger, Comm. Math. Phys. 133 (1990) 163.
- [26] P. Candelas and X. de la Ossa, Nucl. Phys. B355 (1991) 455; P. Candelas, X. de la Ossa, P. S. Green and L. Parkes, Phys. Lett. 258B (1991) 118; Nucl. Phys. B359 (1991) 21.
- [27] R. D'Auria, S. Ferrara and P. Fré, Nucl. Phys. B359 (1991) 705.
- [28] A. C. Cadavid and S. Ferrara, Phys. Lett. 267B (1991) 193; W. Lerche, D. Smit and N. Warner, Nucl. Phys. B372 (1992) 87.
- [29] A. Ceresole, R. D'Auria and T. Regge, Nucl. Phys. B414 (1994) 517; for a recent review, see A. Ceresole, R. D'Auria, S. Ferrara, W. Lerche, J. Louis and T. Regge, *Picard-Fuchs Equations, Special Geometry and Target Space Duality* , preprint CERN-TH.7055/93, POLFIS-TH.09/93, to appear on *Essays on Mirror Manifolds* , vol. II, S. T. Yau editor, International Press (1994).

- [30] A. Ceresole, R. D’Auria, S. Ferrara and A. Van Proeyen, *Duality transformations in supersymmetric Yang–Mills theories coupled to supergravity*, preprint CERN-TH 7547/94, to appear.
- [31] E. Cremmer, S. Ferrara, L. Girardello and A. Van Proeyen, Nucl. Phys. B212 (1983) 413.
- [32] B. de Wit, F. Vanderseypen and A. Van Proeyen, Nucl. Phys. B400 (1993) 463.
- [33] A. Das, Phys. Rev. D15 (1977) 2805; E. Cremmer, J. Scherk and S. Ferrara, Phys. Lett. 68B (1977) 234; E. Cremmer and J. Scherk, Nucl. Phys. B127 (1977) 259.
- [34] E. Cremmer, J. Scherk and S. Ferrara, Phys. Lett. 74B (1978) 61.
- [35] R. Kallosh, A. Linde, T. Ortín, A. Peet and A. Van Proeyen, Phys. Rev. D46 (1992) 5278.
- [36] E. S. Fradkin, M. A. Vasiliev, Nuovo Cim. Lett. **25** (1979) 79; B. de Wit and J. W. van Holten, Nucl. Phys. B155 (1979) 530; B. de Wit, J. W. van Holten and A. Van Proeyen, Nucl. Phys. B167 (1980) 186.
- [37] M. Villasante, Phys. Rev. D45 (1992) 831.
- [38] K. Narain, Phys. Lett. 169B (1986) 41; K. Narain, M. Sarmadi and E. Witten, Nucl. Phys. B279 (1987) 369.
- [39] A. Shapere and F. Wilczek, Nucl. Phys. B320 (1989) 669.
- [40] I. Antoniadis, C. Bachas and C. Kounnas, Nucl. Phys. B289 (1987) 87; H. Kawai, D. C. Lewellen and S. H. Tye, Nucl. Phys. B288 (1987) 1.
- [41] S. Ferrara, C. Kounnas, D. Lust and F. Zwirner, Nucl. Phys. B365 (1991) 431.
- [42] L. J. Dixon, V. S. Kaplunovsky and J. Louis, Nucl. Phys. B355 (1990) 27.
- [43] G. L. Cardoso and B. A. Ovrut, Nucl. Phys. B369 (1992) 351.
- [44] J. P. Derendinger, S. Ferrara, C. Kounnas and F. Zwirner, Nucl. Phys. B372 (1992) 145.
- [45] I. Antoniadis, E. Gava, K. S. Narain and T. R. Taylor, Nucl. Phys. B413 (1994) 162.
- [46] P. Fré and P. Soriani, Nucl. Phys. B371 (1992) 659.
- [47] R. Dijkgraaf, E. Verlinde, H. Verlinde, Comm. Math. Phys. 115 (1988) 649.

- [48] M. K. Prasad and C. M. Sommerfield, Phys. Rev. Lett. 35 (1975) 760; E. B. Bogomol'nyi, Sov. J. Nucl. Phys. 24 (1976) 449; E. Witten and D. Olive, Phys. Lett. 78B (1978) 97; C. Montonen and D. Olive, Phys. Lett. 72B (1977) 117; P. Goddard, J. Nuyts and D. Olive, Nucl. Phys. B125 (1977) 1.
- [49] C. M. Hull and P. K. Townsend, *Unity of Superstring Dualities*, preprint QMW 94-30, hep-th/9410167.
- [50] G. Gibbons and C. Hull, Phys. Lett. B109 (1982) 190; G. Gibbons and N. Manton, Nucl. Phys. B274 (1986) 183.
- [51] A. Sen, Int. J. Mod. Phys. A9 (1994) 3707, hep-th/9402002 and references there in.
- [52] J. Harvey and J. Liu, Phys. Lett. B268 (1991) 40.
- [53] R.R. Khuri, Phys. Lett. B259 (1991) 261; Phys. Lett. B294 (1992) 325.
- [54] R. Kallosh and T. Ortin, Phys. Rev. D48 (1993) 742, hep-th/9302109.